Journal of Global Optimization (2005) 32: 485–494
DOI 10.1007/s10898-003-2685-0

Existence of Solutions and Star-shapedness in Minty Variational Inequalities

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(Received 15 August 2003; accepted 19 August 2003)

Abstract. Minty Variational Inequalities (for short, Minty VI) have proved to characterize a kind of equilibrium more qualified than Stampacchia Variational Inequalities (for short, Stampacchia VI). This conclusion leads to argue that, when a Minty VI admits a solution and the operator F admits a primitive f (that is $F=f'$), then f has some regularity property, e.g. convexity or generalized convexity. In this paper we put in terms of the lower Dini directional derivative a problem, referred to as Minty VI(f'_-, K), which can be considered a nonlinear extension of the Minty VI with $F=f'$ (K denotes a subset of $\mathbb{R}^n$). We investigate, in the case that K is star-shaped, the existence of a solution of Minty VI(f'_-, K) and increasing along rays starting at x^* property of (for short, $f \in \mathrm{IAR}(K, x^*)$). We prove that Minty VI(f'_-, K) with a radially lower semicontinuous function f has a solution $x^* \in \ker K$ if and only if $f \in \mathrm{IAR}(K, x^*)$. Furthermore we investigate, with regard to optimization problems, some properties of increasing along rays functions, which can be considered as extensions of analogous properties holding for convex functions. In particular we show that functions belonging to the class $\mathrm{IAR}(K, x^*)$ enjoy some well-posedness properties.

Key words: existence of solutions, generalized convexity, Minty variational inequality, star-shaped sets, well-posedness

1. Introduction

Variational Inequalities provide a very general and suitable mathematical model for a wide range of problems, in particular equilibrium problems (Baiocchi and Capelo (1984); Kinderlehrer and Stampacchia (1980); Stampacchia (1960)). Minty Variational Inequalities (for short, Minty VI) (Minty (1967)), is the problem of finding a vector $x^* \in K$, such that:

$$\text{Minty VI}(F, K)\ \ \langle F(y), x^* - y\rangle \leqslant 0, \ \ \forall y \in K$$

where $F:\mathbb{R}^n \to \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ is nonempty and $\langle \cdot, \cdot \rangle$ denotes the inner product defined on $\mathbb{R}^n$. In particular the case where the function F has a primitive

$f:\mathbb{R}^n \to \mathbb{R}$, defined (and differentiable) on an open set containing K (i.e. the problem Minty VI(f', K)) has been widely studied, mainly in relation with the minimization of the function f over the set K (see e.g. (Kinderlehrer and Stampacchia (1980))). In (Giannessi (1997)) a vector extension of Minty VI(f', K) is introduced and related to optimality.

Throughout the paper f denotes a real function defined on an open set containing K. For such a function, the lower Dini directional derivative of f at the point $x \in K$ in the direction $u \in \mathbb{R}^n$ is defined as an element of $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ by:

$$f'_-(x, u) = \liminf_{t \to +0} \frac{f(x+tu) - f(x)}{t}.$$

Now we introduce the following problem:

$$\text{Minty VI}(f'_-, K) \;\; f'_-(y, x^* - y) \leqslant 0, \;\; \forall y \in K.$$

The problem is to find $x^* \in K$ for which the inequalities in Minty VI(f'_-, K) are satisfied. This problem obviously reduces to Minty VI(f', K) when f is differentiable on an open set containing K.

The main result of the paper is that, when K is star-shaped, $x^* \in \ker K$ and f is radially lower semicontinuous in K on the rays starting at x^*, the point x^* is a solution of Minty VI(f'_-, K) if and only if f is increasing along such rays (for short, IAR). This condition means that the level sets of f are star-shaped and can be regarded as a convexity-type condition (recall for comparison that, by definition, a function is quasi-convex if and only if its level sets are convex). Therefore we see that IAR functions naturally arise when dealing with Minty variational inequalities.

Moreover, we show that the class of IAR functions has relevant properties with regard to optimization problems and as it happens for convex functions, relations with well-posedness can be established. The latter allows to argue that, when Minty VI(f', K) has a solution (or more generally Minty VI(f'_-, K) is solvable), the primitive optimization problem has some well-posedness property.

2. The Class of IAR Functions

In this section we recall the notion of IAR function and we investigate some basic properties of this class of functions. Such properties can be viewed as extensions of analogous properties holding for convex functions.

DEFINITION 1.

(i) Let K be a nonempty subset of $\mathbb{R}^n$. The set $\ker K$ consisting of all $x \in K$ such that $(y \in K, t \in [0, 1]) \Longrightarrow x + t(y - x) \in K$ is called the kernel of K.
(ii) A nonempty set K is star-shaped if $\ker K \neq \emptyset$.

In the following we will use the abbreviation st-sh for the word star-shaped. It is known (see e.g. (Rubinov (2000))) that the set $\ker K$ is convex for an arbitrary st-sh set K. We will assume, by definition, that the empty set is st-sh.

DEFINITION 2. A function f defined on $\mathbb{R}^n$ is called increasing along rays at a point x^* (for short, $f \in \mathrm{IAR}(x^*)$) if the restriction of this function on the ray $\mathbb{R}_{x^*,x} = \{x^* + \alpha x | \alpha \geqslant 0\}$ is increasing for each $x \in \mathbb{R}^n$. (A function g of one real variable is called increasing if $t_2 \geqslant t_1$ implies $g(t_2) \geqslant g(t_1)$.)

DEFINITION 3. Let $K \subseteq \mathbb{R}^n$ be a st-sh set and $x^* \in \ker K$. A function f defined on K is called increasing along rays at x^* (for short, $f \in \mathrm{IAR}(K, x^*)$), if the restriction of this function on the intersection $\mathbb{R}_{x^*,x} \cap K$ is increasing, for each $x \in K$.

When $n = 1$, $f \in \mathrm{IAR}(K, x^*)$ if and only if it is quasi-convex with a global minimum over K at x^*. The following example shows that when $n \geqslant 2$ and K is a convex set, the class of functions $f \in \mathrm{IAR}(K, x^*)$ is broader then the class of quasi-convex functions with a global minimum at x^*.

EXAMPLE 1. Let $f(x_1, x_2) = x_1^2 x_2^2$ and $K = \mathbb{R}^2$. Then, for $x^* = (0, 0)$ it is easily seen that $f \in \mathrm{IAR}(K, x^*)$, but f is not quasi-convex.

We consider the following problem:

$$P(f, K) \;\; \min f(x), \quad x \in K \subseteq \mathbb{R}^n.$$

A point $x^* \in K$ is a (global) solution of $P(f, K)$ when $f(x) - f(x^*) \geqslant 0$, $\forall x \in K$. The solution is strict if $f(x) - f(x^*) > 0$, $\forall x \in K \setminus \{0\}$. We will denote by $\mathrm{argmin}(f, K)$ the set of solutions of $P(f, K)$. Local solutions of $P(f, K)$ have a clear definition ad we omit it.

The next results give some basic properties of functions which are increasing along rays.

PROPOSITION 1. *Let* $K \subseteq \mathbb{R}^n$ *be a* st-sh *set,* $x^* \in \ker K$ *and* $f \in \mathrm{IAR}(K, x^*)$. *Then:*

(i) x^* *is a solution of* $P(f, K)$*;*
(ii) *No point* $x \in K$, $x \neq x^*$, *can be a strict local solution of* $P(f, K)$.
(iii) $x^* \in \ker \arg\min(f, K)$.

Proof.

(i) Let $x \in K$ and set $z(t) = x^* + t(x - x^*)$, $t \in [0, 1]$. Since $x^* \in \ker K$, then $z(t) \in K$, $\forall t \in [0, 1]$ and since $f \in \text{IAR}(K, x^*)$, we have $f(z(t)) \geqslant f(x^*) = f(z(0))$, $\forall t \in [0, 1]$ and in particular $f(z(1)) = f(x) \geqslant f(x^*)$. Since $x \in K$ is arbitrary, then x^* is a global minimizer of f over K.

(ii) Let x and $z(t)$ as above. Since $f \in \text{IAR}(K, x^*)$, it easily follows $f(z(t)) \leqslant f(x) = f(z(1))$, $\forall t \in [0, 1]$. If U is an arbitrary neighborhood of x, then for t 'near enough' to 1, we have $z(t) \in U$ and so x cannot be a strict local minimizer for f over K.

(iii) Let $x \in \text{argmin}(f, K), x \neq x^*$. Since $z(t) \in K$, we have $f(z(t)) \leqslant f(x)$, $\forall t \in [0, 1]$ and readily follows that for every $t \in [0, 1], z(t) \in \text{argmin}(f, K)$.

The next Proposition can be found in (Zaffaroni (2001)).

PROPOSITION 2. *Let $K \subseteq \mathbb{R}^n$ be a* st-sh *set, $x^* \in \ker K$ and f be a function defined on K. Then $f \in \text{IAR}(K, x^*)$ if and only if for each $c \in R$ with $c \geqslant f(x^*)$, we have $x^* \in \ker\ \text{lev}_{\leqslant c} f$.*

3. Minty Variational Inequalities and IAR Functions

In this section we prove that a radially lower semicontinuous function f belongs to the class $\text{IAR}(K, x^*)$ if and only if x^* solves Minty $\text{VI}(f'_-, K)$.

DEFINITION 4. Let $K \subseteq \mathbb{R}^n$, $x^* \in \ker K$ and let f be a function defined on an open set containing K. The function f is said to be radially lower semicontinuous in K along rays starting at x^*, if for each $x \in K$, the restriction of f on the interval $\mathbb{R}_{x^*,x} \cap K$ is lower semicontinuous.

We will use the abbreviation $f \in \text{RLSC}(K, x^*)$ to denote that f satisfies the previous definition.

THEOREM 1 (Mean value theorem). *Let $x^* \in \ker K$, $f \in \text{RLSC}(K, x^*)$, $y \in K$, and $t > 0$ such that $y + t(x^* - y) \in K$. Then there exists a number $\alpha \in]0, t]$, such that:*

$$f(y + t(x^* - y)) - f(y) \leqslant t f'_-(y + \alpha(x^* - y), x^* - y).$$

Proof. Let $h(s) = f(y + s(x^* - y)) - \frac{s}{t}[f(y + t(x^* - y)) - f(y)]$. Then the mean value inequality is equivalent to the existence of a number $\alpha \in]0, t]$ such that:

$$h_{-}(\alpha) := \liminf_{r \to +0} \frac{h(\alpha + r) - h(\alpha)}{r}.$$

Clearly we can write $y + s(x^* - y) = x^* + (1 - s)(y - x^*)$ and hence h is lower semicontinuous. According to Weierstrass Theorem, it attains its global minimum at some point $\hat{t} \in [0, t]$. Indeed we have $h(0) = h(t) = f(y)$ and therefore if the global minimum is achieved for $\hat{t} = 0$ it is also achieved for $\hat{t} = t$. Hence, for $\alpha = \hat{t}$ we have $h_{-}(\alpha) \geqslant 0$ and the Theorem is proved.

THEOREM 2. *Let* $K \subseteq \mathbb{R}^n$ *be a* st-sh *set and* $x^* \in kerK$.

(i) *If* x^* *solves* Minty VI(f'_{-}, K) *and* $f \in$ RLSC(K, x^*)*, then* $f \in IAR(K, x^*)$.
(ii) *Conversely, if* $f \in$ IAR(K, x^*)*, then* x^* *is a solution of* Minty VI(f'_{-}, K).

Proof. (i) Let x^* be a solution of Minty VI(f$'_{-}$,K), $y \in K$ and $y + t_2(x^* - y), y + t_1(x^* - y)$ be points in $\mathbb{R}_{x^*,x} \cap K$, with $t_2 > t_1 \geqslant 0$. Applying the previous Theorem we have:

$$\begin{aligned} &f(y + t_2(x^* - y)) - f(y + t_1(x^* - y)) \\ &\quad \leqslant (t_2 - t_1) f'_{-}(y + \alpha(x^* - y), x^* - y) \leqslant 0, \end{aligned}$$

with $\alpha \in]t_1, t_2]$. It is easily seen that this proves $f \in$ IAR(K, x^*)

(ii) Assume that $f \in$ IAR(K, x^*) and let $y \in K$. For every $t \in [0, 1]$, we have: $f(y + t(x^* - y)) = f(x^* + (1 - t)(y - x^*)) \leqslant f(y)$ and hence:

$$\frac{f(y + t(x^* - y)) - f(y)}{t} \leqslant 0.$$

Taking lim inf as $t \to +0$, we obtain that x^* solves Minty VI(f$'_{-}$, K).

In the previous Theorem the assumption $f \in$ RLSC(K, x^*) appears in only one of the two opposite implications. A natural question arises, whether it cannot be dropped at all. The next examples give a negative answer to this question.

EXAMPLE 2. Let $K = \mathbb{R}, x^* = 0$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \begin{cases} 1, & \text{if} x \neq 2 \\ 3, & \text{if} x = 2 \end{cases}$$

Then $f \notin$ RLSC(k, x^*) and it holds $f'_{-}(y, x^* - y) \leqslant 0, \forall y \in \mathbb{R}$, but $f \notin$ IAR(K, x^*).

EXAMPLE 3. Define the function $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=0$ for $x=0$ or x irrational, and $f(x)=-q$ for $x\neq 0$ rational with $x=p/q, q>0$ and p and q mutually prime. The function f is not lower semicontinuous and the Dini derivatives are $f'_-(x,u)=-\infty$ for each $x\in\mathbb{R}$ and $u\in\mathbb{R}\backslash\{0\}$. Consequently, each point $x^*\in\mathbb{R}$ is a solution of Minty VI(f'_-,K). At the same time f has no global minimizers. In particular $x^*=0$ is among the solutions of Minty VI(f'_-,K), which is a global maximizer of f. Even more, while having no global minimizers, there is a dense in $\mathbb{R}$ set of points, namely the set of the irrational numbers, each of which is both a solution of Minty VI(f'_-,K) and a global maximizer of f.

COROLLARY 1. *Let* $x^*\in\ker K$ *and let* $f\in \mathrm{RLSC}(K,x^*)$. *If* x^* *solves* MintyVI(f'_-,K), *then* x^* *solves* $P(f,K)$.

Proof. It is immediate from the previous Theorem and Proposition 1.

REMARK 1. The previous Corollary extends a classical result which states that if K is a convex set, any solution of Minty VI(f',K) solves $P(f,K)$.

COROLLARY 2. *If* $x^*\in\ker K$ *and f is differentiable on an open set containing K, then* x^* *solve Minty VI*(f',K) *if and only if* $f\in\mathrm{IAR}(K,x^*)$.

4. Minty Variational Inequalities and Well-posedness

In this section we show that functions $\in\mathrm{IAR}(K,x^*)$ enjoy some well-posedness properties, analogously to convex functions.

DEFINITION 5.

(i) A sequence $x^k\in K$ is a minimizing sequence for $P(f,K)$, when $f(x^k)\to\inf_K f(x)$.

(ii) A sequence x^k is a generalized minimizing sequence for $P(f,k)$ when:

$$f(x^k)\to\inf_K f(x),\quad \mathrm{dist}(x^k,K)\to 0$$

(here dist(x,K) denotes the distance from the point x to the set K.)

DEFINITION 6.

(i) Problem $P(f,K)$ is Tykhonov well-posed When it admits a unique solution x^* and every minimizing sequence for $P(f,K)$ converges to x^*.

(ii) Problem $P(f, K)$ is Levitin-Polyak well-posed when it admits a unique solution x^* and every generalized minimizing sequence for $P(f, K)$ converges to x^*.

Let us denote with argmin(f, k) the set of solutions of $P(f, K)$ and consider the sets:

$$L^f(\varepsilon) := \{x \in K \mid f(x) \leqslant \inf_K f(x) + \varepsilon\}$$

and

$$L_s^f(\varepsilon) := \{x \in K \mid \text{dist}(x, K) \leqslant \varepsilon, f(x) \leqslant \inf_K f(x) + \varepsilon\}.$$

We recall the following result (see e.g. (Dontchev and Zolezzi (1993))).

THEOREM 3.

(i) *If $P(f, K)$ is Tykhonov well-posed, then diam $L^f(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

(ii) *Let f be lower semicontinuous and bounded from below on K. If $\lim_{\varepsilon \to 0} \text{diam} L^f(\varepsilon) = 0$, then $P(f, K)$ is Tykhonov well-posed.*

(iii) *If K is closed and f is lower semicontinuous and bounded from below on K, then diam $L_s^f(\varepsilon) \to 0$ as $\varepsilon \to 0$ implies that P(f,K) is Levitin-Polyak well-posed.*

DEFINITION 7. Problem $P(f, K)$ is said Tykhonov well-posed in the generalized sense when argmin$(f, K) \neq \emptyset$ and every minimizing sequence for $P(f, K)$ has some subsequence that converges to an element of argmin(f, K).

Of course, $P(f, K)$ is Tykhonov well-posed if and only if argmin(f, K) is a singleton and $P(f, K)$ is well posed in the generalized sense.

DEFINITION 8. Problem $P(f, K)$ is stable when argmin $(f, K) \neq \emptyset$ and for every sequence x^k minimizing for $P(f, K)$ we have:

$$\text{dist}[x^k, \text{argmin}(f, K)] \to 0.$$

The following results extend to IAR functions some classical well-posedness properties of convex functions.

THEOREM 4. *Let K be a closed subset of $\mathbb{R}^n$, $x^* \in \ker K$ and let $f \in \text{IAR}(K, x^*)$ be a lower semicontinuous function. If argmin (f, K) is bounded, then P(f,K) is stable.*

Proof. Let $x^k \in K$ be a minimizing sequence for $P(f, K)$, but, by contradiction, assume that $\text{dist}[x^k, \text{argmin}(f, K)] \nrightarrow 0$. Then, for infinitely many k we have:

$$x^k \notin \text{argmin}(f, K) + \delta B,$$

for some positive δ (here B denotes the open unit ball in $\mathbb{R}^n$). Without loss of generality, we can assume that this holds for every k. If x^k is a bounded sequence, one can think that x^k converges to a point $\bar{x} \notin \text{argmin}(f, K)$, but this is absurdo.

We shall therefore assume x^k is unbounded. If this holds, $\forall k$ there exists $t_k \in (0, 1)$ such that $y^k = t_k x^* + (1 - t_k)x^k \in$ bd [argmin $(f, K) + \delta B$] (here bd A denotes the boundary of the set A). Since argmin(f, K) is bounded and K is closed, one can think that $y^k \to \bar{y} \in K$ with $\bar{y} \notin \text{argmin}(f, K)$. Hence $\forall \varepsilon > 0$ and for k 'large enough', since $f \in \text{IAR}(K, x^*)$, we get:

$$f(x^*) \leqslant f(y^k) \leqslant f(x^k) \leqslant \inf_K f(x) + \varepsilon = f(x^*) + \varepsilon$$

and the lower semicontinuity of f gives the absurdo $f(x^*) = f(\bar{y})$.

COROLLARY 3. *Let K be a closed subset of* $\mathbb{R}^n, x^* \in$ *ker* $K, f \in$ IAR (K, x^*) *be lower semi and* argmin (f, K) *be compact. Then $P(f, K)$ is Tykhonov well posed in the generalized sense.*

Proof. It easily follows observing that when argmin (f, K) is compact, then stability is equivalent to Tykhonov generalized well-posedness.

COROLLARY 4. *Let K be a closed subset of* $\mathbb{R}^n, x^* \in \ker K$ *and* $f \in$ IAR (K, x^*) *be lower semicontinuous. If* argmin (f, K) *is a singleton, then $P(f, K)$ is Tykhonov well-posed.*

The assumption that argmin (f, K) is bounded is essential to prove Theorem 4, as it is shown in the following example.

EXAMPLE 4. Let $K = \mathbb{R}^2_+, x^* = (0, 0)$ and consider the function therein defined:

$$f(x, y) = \begin{cases} 0, & \text{if}\,(x, y) = (0, y) \\ t(\bar{x} - 1)^2, & \text{if}\,(x, y) = \left(t\bar{x}, t\frac{2-\bar{x}}{\bar{x}-1}\right), \quad \text{for} \;\; t > 0, 1 < \bar{x} \leqslant 2. \end{cases}$$

Clearly, $f \in$ IAR (K, x^*) and argmin $(f, K) = \{(x, y) \in \mathbb{R}^2_+ | x = 0\}. P(f, K)$ is not stable, since the sequence $(1 + \frac{1}{n}, n - 1)$ is minimizing, but its distance from argmin (f, K) does not tend to 0.

LEMMA 1. *Let* $x^* \in \ker K$. *Then dist* $(\cdot, K) \in \text{IAR}(x^*)$.

Proof. Without loss of generality we assume $x^* = 0$. Consider a point $x \in \mathbb{R}^n$ and two positive scalars t_1, t_2, with $t_2 \geqslant t_1$ and set:

$$\text{dist}(t_2 x, K) = \inf_{y \in K} \|y - t_2 x\| = l.$$

Consider a sequence $y^k \in K$, such that $\|y^k - t_2 x\| \leqslant l + 1/k$. Since $\frac{t_1}{t_2} y^k \in K$, we have:

$$\text{dist}(t_1 x, K) \leqslant \frac{t_1}{t_2} \|y^k - t_2 x\| \leqslant \frac{t^1}{t_2}(l + 1/k) \leqslant l + 1/k$$

and for $k \to +\infty$ we get $\text{dist}(t_1 x, K) \leqslant l$.

THEOREM 5. *Assume that K is a closed set*, $x^* \in \ker K$, *f is a lower semicontinuous continuous function and there exists* $\tau > 0$ *such that* $f \in \text{IAR}(K_\tau, x^*)$, *where* $K_\tau = K + \tau B$. *If P(f, K) is Tykhonov well-posed, then diam* $L_s^f(\varepsilon) \to 0$, *as* $\varepsilon \to 0$.

Proof. Ab absurdo, assume that diam $L_s^f(\varepsilon) \not\to 0$. Hence there exists a positive number δ such that $\forall \varepsilon > 0$ one can find a point $x(\varepsilon)$ with $\text{dist}(x, K) \leqslant \varepsilon$ and $f(x(\varepsilon)) \leqslant f(x^*) + \varepsilon$, but $x(\varepsilon) \notin x^* + \delta B$. Let $\varepsilon = \frac{1}{k}$, $x^k := x(\varepsilon)$ and assume first that x^k is bounded. Hence we can assume that x^k converges to some $\bar{x}$. Since $\text{dist}(x^k, K) \leqslant \frac{1}{k}$ and K is closed, then $\bar{x} \in K$. Furthermore we have $f(x^k) \leqslant f(x^*) + \frac{1}{k}$ and recalling that f is lower semicontinuous and that x^* minimizes f over K, we get $f(\bar{x}) = f(x^*)$, which contradicts the assumption of Tykhonov well-posedness.

Let assume, therefore, x^k is unbounded. Hence for k 'large enough', $x^k \in K_\tau$ and we can find $\delta > 0$ such that $x^k \notin x^* + \delta B$. Let now $y^k = t_k x^k + (1 - t_k)x^* \in \text{bd}(x^* + \delta B)$, for $t \in (0, 1)$. Since $x^* \in \ker K$, then dist $(\cdot, K) \in \text{IAR}(x^*)$ and from:

$$\text{dist}(y^k, K) \leqslant \text{dist}\,(x^k, K) \leqslant \frac{1}{k},$$

we get dist $(y^k, K) \to 0$. Since $f \in \text{IAR}(K_\tau, x^*)$, for k 'large enough' we have:

$$f(x^*) \leqslant f(y^k) \leqslant f(x^k) \leqslant f(x^*) + \frac{1}{k}$$

and hence y^k is a generalized minimizing sequence. Now the well-posedness is contradicted since we can assume $y^k \to \bar{y} \in K$, $\bar{y} \neq x^*$ and the lower semicontinuity of f implies $f(\bar{y}) = f(x^*)$.

COROLLARY 5. *Under the hypotheses of Theorem 5, $P(f, K)$ is Levitin-Polyak well-posed.*

Proof. It follows immediately from iii) of Theorem 3.

Acknowledgments

The authors are thankful to Professor Franco Giannessi, whose observations originated this work and to Professor Alex Rubinov, for his valuable comments and suggestions which led to the final version.

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